Linear Programming

Lecture 20

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CS3000 Algorithms and Data

Linear Programs

LP Duality

1. Linear Programs

Linear Programs (LPs)

- A optimization problem where we aim to maximize/minimize a linear objective.
- All the constraints in the problem are also linear.

$$
\begin{aligned}
\max \quad & c^T x \\
\text{such that} \quad & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{such that} \quad & \sum_{j=1}^{m} a_{ij} x_j \leq b_i && \text{for all } 1 \leq i \leq m \\
& x_j \geq 0 && \text{for all } 1 \leq j \leq n
\end{aligned}
$$

Brewery Problem

Brewery Problem: How much ale and beer should be produced to maximize profit?

Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Total	480	160	1190	

- All ale (34 units) $\implies 442$
- All beer (32 units) $\implies 736$
- 7.5 units ale, 29.5 units beer $\implies 776$
- 12 units ale, 28 units beer $\implies 800$

Brewery Problem Modeled as an LP

$$
\begin{aligned}
\max \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \leq 480 \\
& 4A + 4B \leq 160 \\
& 35A + 20B \leq 1190 \\
& A, B \geq 0
\end{aligned}
$$

Entrepreneur Problem

Entrepreneur Problem: Buy raw materials at minimum cost from the brewer

Beverage	Corn	Hops	Malt	Profit
Ale	5	4	35	13
Beer	15	4	20	23
Total	480	160	1190	

Let C, H, M are units of corn, hops, malt.

Brewer won't sell if, for example, $5C + 4H + 35M \leq 13$.

Entrepreneur Problem Modeled as an LP

$$
\begin{aligned}
\min \quad & 480C + 160H + 1190M \\
\text{s.t.} \quad & 5C + 4H + 25M \geq 13 \\
& 15C + 4H + 20M \geq 23 \\
& C, H, M \geq 0
\end{aligned}
$$

2. LP Duality

LP Duality

The "primal" formulation of an LP

$$\begin{aligned} \max \quad & c^T x \\ \text{such that} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

has a "dual" formulation

$$\begin{aligned} \min \quad & b^T y \\ \text{such that} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

- Just like max flow-min cut, the max of the primal is upper bounded by min of dual.
- In fact, the strong duality also holds: $\max c^T x = \min b^T y$. (If feasible.)

Maximum Flow is an LP

Maximum Flow f: variables are $f(e)$, the flow through edge e.

$$
\begin{aligned}
\max \quad & \sum_{e=(s,v)\in E} f(e) & \\
\text{such that} \quad & f(e) \leq c(e) & \forall e \in E \text{ (Capacity bound)} \\
& \sum_{e=(u,v)\in E} f(e) - \sum_{e'=(v,w)\in E} f(e') = 0 & \forall v \in V - \{s,t\} \text{ (Flow Conservation)} \\
& f(e) \geq 0 & \forall e \in E \text{ (Non-negative flow)}
\end{aligned}
$$

Minimum Cut (A, B): the variables are
$s_{uv} \in \{0, 1\}$ edge (u, v) in cut-set or not
$a_u \in \{0, 1\}$ vertex $u \in A$.
Note that by definition $a_s = 1, a_t = 0$.

$$
\begin{aligned}
\min \quad & \sum_{(u,v)\in E} c_{uv} s_{uv} & \\
\text{such that} \quad & s_{uv} - a_u + a_v \geq 0 & \forall (u, v) \in E \text{ (force legal cut)} \\
& s_{uv} \geq 0 & \forall (u, v) \in E \\
& a_v \geq 0 & \forall v \in V - \{s, t\}
\end{aligned}
$$