CS3000 Algorithms

Recitations 01

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Problem 1 (Induction)

a. Prove the following formulas for the sums of the arithmetic and geometric series/progression.

(a) Arithmetic Progression: $\sum_{i=1}^{n} a + (i-1)d = \frac{1}{2}n(a + (a + (n-1)d))$

The summation is half of n times the first term + last term.

(b) Geometric Progression: $\sum_{i=0}^{n-1} ar^i = \frac{a - ar^n}{(1-r)}$, where $r \neq 1$.

The sum is first term - next term divided by 1 - common ratio.

b. (a) **(*)** Prove that the following statements hold using induction; clearly provide the base case as well as the proof of the inductive step.

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+p-1) = \frac{1}{p+1}n(n+1)(n+2)(n+3)\cdots(n+p)$$

(b) **(*)** As a further exercise, try to find formulas for $\sum_{i=1}^{n} i$ (sum of first n natural numbers), $\sum_{i=1}^{n} i^2$ (sum of squares of first n natural numbers), $\sum_{i=1}^{n} i^3$ (sum of cubes of first n natural numbers) using the above formulas.

Problem 2 (Basic Prerequisite Recap)

These are some basic mathematical prerequisites that you should know. They will be covered during the recitation only if time permits.

a. We are going to prove the classic result that $\sqrt{2}$ is irrational. To do this, we use proof by contradiction. Let's call the statement we are trying to prove P. If we want to prove that P is true (in this case, we want to prove that $\sqrt{2}$ is irrational), we make the assumption that $\neg P$ is true. Using this assumption, we try to derive find some other statement C, so that $\neg P \implies C \wedge \neg C$, which is logically impossible (a contradiction). And this means that P has to be true.

First we need this result.

(a) Explain why if n^2 is even, then n is even.

(b) Now prove that $\sqrt{2}$ is irrational.

Note: This can also be proven through the Fundamental Theorem of Arithmetic, where every rational number has a unique decomposition based on prime numbers...

b. (a) How many ways can you rearrange the letters in the word: ALGORITHM?

(b) What about the word: ALGARATHM?

(c) What if no two vowels can appear together (in ALGORITHM)?

(d) What if you cannot have more than two consonants together at a time (in ALGORITHM)?

c. You are picking 3 different numbers from 1 to 15.

   (a) How many ways can you do it such that the product is divisible by either 2 or 3?

   (b) How many ways can you do it such that the product is divisible by 4?

d. Assume that friendship is a symmetric property (if Alice is friends with Bob, then Bob is friends with Alice). Show that in a group of n people, there will always be at least two people with the same number of friends.

e. (a) Prove that $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ by counting a set in two different ways.

   (b) Prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ by counting a set in two different ways.

f. Determine whether or not the expression converges to a value. If it converges, determine its value.

   (a) $\sum_{i=1}^{\infty} \frac{1}{2^i}$ (b) $\sum_{i=1}^{\infty} \frac{1}{i}$ (c) $\sum_{k=0}^{n} \binom{n}{k}$

Problem 3 (Basics of Asymptotics)

Make sure you are familiar with the definitions of the various Big-Oh notations:
$f(n) = o(g(n)); f(n) = O(g(n)); f(n) = \Theta(g(n)); f(n) = \Omega(g(n)); f(n) = \omega(g(n))$

(a) Show that for $a, b \in \mathbb{R}^+$, the following hold:

   a. $n^a = O(n^b)$, whenever $a \leq b$.

   b. $n^a = o(n^b)$, whenever $a < b$.

   c. $n^a + n^b = O(n^b)$, whenever $a \leq b$.

   d. $n^a + n^b = o(n^b)$, whenever $a < b$.[1]

   You could think about similar statements that use ω and Ω instead.

(b) Let $f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$ and $g(n) = n^b$. Using the facts you proved in subpart (a), come up with a value of b in relation to k such that the following hold:

   a. $f(n) = o(g(n))$

   b. $f(n) = O(g(n))$

   c. $f(n) = \omega(g(n))$

   d. $f(n) = \Omega(g(n))$

   e. $f(n) = \Theta(g(n))$

(c) Show that for $a, b, c, d, k \in \mathbb{R}^+$ and $k > 1$, the following holds: $a \ll \log_b^c n \ll n^d \ll k^n \ll n^n$, where we define and use $x \ll y$ as a shorthand notation which denotes that $x = o(y)$.

   Note that here only n is growing to infinity, everything else is a constant. Thus, in words, this is asking you to prove that: Constants grow slower than logarithms, which grow slower than polynomials, which grow slower than constants raised to an exponent, which grow slower than n^n where both the base and the exponent are dependent on n.

Problem 4 (Number theory practice)

Solve the following number theoretic problems:

1. Find the values of the following without explicit calculation:

[1] This is incorrect, but is being kept as is to illustrate how to prove its incorrectness.

- $6^{987382934023} \mod 37$
- $7^{99288399289} \mod 5$
- $13^{12301293120} \mod 17$
- $GCD(21, 28, 49)$
- $7^{920} \pmod 5$
- $6^{273} \pmod{11}$

2. What is the remainder of $1! + 2! + 3! + \dots$ when divided by 9?
3. Prove that $2^n + 6 \cdot 9^n$ is always divisible by 7 for any positive integer n.
4. Let $n \in \mathbb{N}^+$ be an integer not divisible by 17. Prove that 17 divides either $n^8 + 1$ or $n^8 - 1$.
5. Let p be a prime. Prove that $(p-1)! \equiv -1 \pmod p$.
6. Find all positive integers n such that $3^n - n^2$ is divisible by 5.

Problem 5 (Recurrences Practice)

Solve the following recurrences:

1. $T(n) = 3T(n/2) + n^2$
2. $T(n) = 4T(n/2) + n^2$
3. $T(n) = T(n/2) + 2^n$
4. $T(n) = 16T(n/4) + n$
5. $T(n) = 2^n T(n/2) + n^n$
6. $T(n) = 2T(n/2) + n \lg n$
7. $T(n) = 2T(n/2) + n/\log n$
8. $T(n) = 2T(n/4) + n^{0.51}$
9. $T(n) = 0.5T(n/2) + 1/n$
10. $T(n) = 16T(n/4) + n!$
11. $T(n) = \sqrt{2}T(n/2) + \log n$
12. $T(n) = 3T(n/2) + n$
13. $T(n) = 3T(n/3) + \sqrt{n}$
14. $T(n) = 4T(n/2) + cn$
15. $T(n) = 3T(n/4) + n \log n$
16. $T(n) = 3T(n/3) + n/2$
17. $T(n) = 6T(n/3) + n^2 \log n$
18. $T(n) = 4T(n/2) + n/\log n$
19. $T(n) = 7T(n/3) + n^2$
20. $T(n) = 4T(n/2) + \log n$

Problem 6 (Divide and Destroy)

Given a length n array $A[1 \dots n]$, describe an $O(\log n)$ algorithm for the following:

(a) A is a circular sorted array, that is, an array which is sorted and then rotated by k indices such that $A[1 \dots k-1]$ and $A[k \dots n]$ are both sorted and concatenating $A[k \dots n] \circ A[1 \dots k-1]$ would return an overall sorted array. Find the value k by which A was rotated.

(b) A is a unimodal array, that is, an array in which $A[1 \dots k]$ is sorted in ascending order and $A[k \dots n]$ is sorted in descending order so that $A[k]$ forms a unique peak/mode in the array. Find the index of an input number x.

(c) A is a sorted array, where all numbers occur twice except a number x which occurs only once. Find the index of x.

[***Hint**: You've already seen divide and destroy algorithms for searching.*]